Example 7: Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 1 & 1 & 2 \end{bmatrix}$. Expanding along the second column or third row will be the most efficient due to the location of the 0 but many calculations will still be required.

$$|A| = 2(-1)^{3+1}\begin{vmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 1 & 1 & 2 \end{vmatrix} + (-3)(-1)^{3+3}\begin{vmatrix} 2 & 1 & 5 \\ 1 & 1 & -1 \\ 3 & 1 & 2 \end{vmatrix} + 1(-1)^{3+4}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -4 \\ 3 & 1 & 1 \end{vmatrix}$$

> *Theorem 4.3:* Let A in $\mathbb{R}^{n\times n}$.
>
> 1. If B is a matrix obtained by interchanging any two rows or interchanging any two columns of A, then $\det(B) = -\det(A)$.
> 2. If A has two identical rows or columns then $\det(A) = 0$
> 3. If B is a matrix obtained by multiplying a row or a column of A by a scalar k, then $\det(B) = k\det(A)$.
> 4. If B is a matrix obtained from A by adding a multiple of row i to row j or a multiple of column i to column j for $i \neq j$, then $\det(B) = \det(A)$.

Example 8: Again consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 1 & 1 & 2 \end{bmatrix}$. Quickly calculate $\det(A)$ by making strategic use of part 3 of theorem 3.

$$|A| = \begin{vmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 5 \\ -1 & 0 & -5 & -6 \\ 2 & 0 & -3 & 1 \\ 1 & 0 & 0 & -3 \end{vmatrix} = 1(-1)^{1+2}\begin{vmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ 1 & 0 & -3 \end{vmatrix} =$$

$$= -1\left(1\begin{vmatrix} -5 & -3 \\ -6 & 1 \end{vmatrix} + (-3)\cancel{(-1)^{3+3}}\begin{vmatrix} -1 & -5 \\ 2 & -3 \end{vmatrix}\right) = -1\left(-23 - 3(13)\right)$$

$$= 23 + 39 = 62.$$